COGREDIENT STANDARD FORMS OF ORTHOGONAL MATRICES OVER FINITE LOCAL RINGS OF ODD CHARACTERISTIC

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Abstract. In this work, we present a cogredient standard form of an orthogonal space over a finite local ring of odd characteristic.

1. Units and the square mapping

A *local ring* is a commutative ring which has a unique maximal ideal. For a local ring R, we denote its unit group by $R^\times$ and it follows from Proposition 1.2.11 of [1] its unique maximal ideal $M = R \setminus R^\times$ consists of all non-unit elements. We also call the field R/M, the *residue field of* R.

Example 1. If p is a prime, then $\mathbb{Z}_{p^n}$, $n \in \mathbb{N}$, is a local ring with maximal ideal $p\mathbb{Z}_{p^n}$ and residue field $\mathbb{Z}_{p^n}/p\mathbb{Z}_{p^n}$ isomorphic to $\mathbb{Z}_p$. Moreover, every field is a local ring with maximal ideal $\{0\}$.

Recall a common theorem about local rings that:

Theorem 1.1. *Let R be a local ring with unique maximal ideal M. Then $1 + m$ is a unit of R for all $m \in M$. Furthermore, $u + m$ a unit in R for all $m \in M$ and $u \in R^\times$.*

Proof. Suppose that $1 + m$ is not a unit. Since R is local, $1 + m \in M$. Hence, 1 must be in M, which is a contradiction. Finally, we note that $u + m = u(1 + u^{-1}m)$ is a unit in R. □

Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field k. Then R is of order an odd prime power, and so is M. From Theorem XVIII. 2 of [3] we have that the unit group of R, denoted by $R^\times$, is isomorphic to $(1 + M) \times k^\times$. Consider the exact sequence of groups

$$1 \longrightarrow K_R \longrightarrow R^\times \longrightarrow (R^\times)^2 \longrightarrow 1$$

where $\theta : a \longmapsto a^2$ is the square mapping on $R^\times$ with kernel $K_R = \{a \in R^\times : a^2 = 1\}$ and $(R^\times)^2 = \{a^2 : a \in R^\times\}$. Note that K_R consists of the identity and all elements of order two in $R^\times$. Since R is of odd characteristic and $k^\times$ is cyclic, $K_R = \{\pm 1\}$. Hence, $[R^\times : (R^\times)^2] = |K_R| = 2$.

Proposition 1.2. *Let R be a finite local ring of odd characteristic with unique maximal ideal M and residue field k.*

(1) The image $(R^\times)^2$ is a subgroup of $R^\times$ with index $\left[R^\times : (R^\times)^2\right] = 2$.
(2) For $z \in R^\times \setminus (R^\times)^2$, we have $R^\times \setminus (R^\times)^2 = z(R^\times)^2$ and $|(R^\times)^2| = |z(R^\times)^2| = (1/2)|R^\times|$.
(3) For $u \in R^\times$ and $a \in M$, there exists $c \in R^\times$ such that $c^2(u + a) = u$.
(4) If $-1 \notin (R^\times)^2$ and $u \in R^\times$, then $1 + u^2 \in R^\times$.
(5) If $-1 \notin (R^\times)^2$ and $z \in R^\times \setminus (R^\times)^2$, then there exist $x, y \in R^\times$ such that $z = (1 + x^2)y^2$.

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Proof. We have proved (1) in the above discussion and (2) follows from (1). Let $u \in R^\times$ and $a \in M$. Then $u^{-1}(u+a) = 1 + u^{-1}a \in 1 + M$, so $(u^{-1}(u+a))^{|1+M|+1} = u^{-1}(u+a)$. Since $|1+M| = |M|$ is odd, $u^{-1}(u+a) = (c^{-1})^2$ for some $c \in R^\times$. Thus, $c^2(u+a) = u$ which proves (3).

For (4), assume that $-1 \notin (R^\times)^2$ and let $u \in R^\times$. Suppose that $1 + u^2 = x \in M$. Then $u^2 = -(1-x)$. Since $|M|$ is odd and $1-x \in 1+M$, $(u^{|M|})^2 = (-(1-x))^{|M|} = (-1)^{|M|}(1-x)^{|M|} = (-1)(1) = -1$, which contradicts -1 is non-square. Hence, $1 + u^2 \in R^\times$.

Finally, we observe that $|1 + (R^\times)^2| = |(R^\times)^2|$ is finite. If $1 + (R^\times)^2 \subseteq (R^\times)^2$, then they must be equal, so there exists $b \in (R^\times)^2$ such that $1 + b = 1$, which forces $b = 0$, a contradiction. Hence, there exists an $x \in R^\times$ such that $1 + x^2 \notin (R^\times)^2$. By (4), $1 + x^2 \in R^\times$. Therefore, for a non-square unit z, we have $R^\times$ is a disjoint union of cosets $(R^\times)^2$ and $z(R^\times)^2$, so $1+x^2 = z(y^{-1})^2$ for some $y \in R^\times$ as desired. □

In what follows, we shall apply the above proposition to obtain a nice cogredient standard form of an orthogonal space over a finite local ring of odd characteristic. This work generalizes the results over a Galois ring studied in [2].

2. Cogredient standard forms of orthogonal spaces

Throughout this section, we let R be a finite local ring of odd characteristic.

Notation. *For any $l \times n$ matrix A and $q \times r$ matrix B over R, we write*

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

which is an $(l+q) \times (n+r)$ matrix over R.

For any matrices $S_1, S_2 \in M_n(R)$, if there exists an invertible matrix P such that $PS_1P^T = S_2$, we say that S_1 is *cogredient* to S_2 over R and we write $S_1 \approx S_2$. Note that $S \approx c^2 S$ for all $c \in R^\times$. The next lemma is a key for our structure theorem.

Lemma 2.1. *For a positive integer ν and $z \in R^\times \setminus (R^\times)^2, zI_{2\nu}$ is cogredient to $I_{2\nu}$.*

Proof. If $-1 = u^2$ for some $u \in R^\times$, we may choose $P = 2^{-1}\begin{pmatrix} (1+z) & u^{-1}(1-z) \\ u(1-z) & (1+z) \end{pmatrix}$ whose determinant is $z \in R^\times$. Note that our R of odd characteristic, so 2 is a unit. Hence, P is invertible and $PP^T = zI_2$. Next, we assume that -1 is non-square. Then, by Proposition 1.2 (5), $z = (1+x^2)y^2$ for some units x and y in $R^\times$. Choose $Q = \begin{pmatrix} xy & y \\ -y & xy \end{pmatrix}$. Then $\det Q = (1+x^2)y^2 = z \in R^\times$, so Q is invertible and $QQ^T = \begin{pmatrix} (1+x^2)y^2 & 0 \\ 0 & (1+x^2)y^2 \end{pmatrix} = zI_2$. Therefore, $zI_{2\nu} = \overbrace{zI_2 \oplus \cdots \oplus zI_2}^{\nu\text{'s}}$ is cogredient to $I_{2\nu} = \overbrace{I_2 \oplus \cdots \oplus I_2}^{\nu\text{'s}}$. □

Let R be a local ring. Let V be a free R-module of rank n, where $n \geq 2$. Assume that we have a function $\beta : V \times V \to R$ which is R-bilinear, symmetric and the R-module morphism from V to $V* = \hom_R(V, R)$ given by $\vec{x} \mapsto \beta(\cdot, \vec{x})$ is an isomorphism. For $\vec{x} \in V$, we call $\beta(\vec{x}, \vec{x})$ the *norm of* $\vec{x}$. The pair (V, β) is called an *orthogonal space*. Moreover, if $\beta = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis of V, then the associated matrix $[\beta]_{\mathcal{B}} = [\beta(\vec{b}_i, \vec{b}_j)]_{n \times n}$. We say that $\mathcal{B}$ is an orthogonal basis if $\beta(\vec{b}_i, \vec{b}_i) = u_i \in R^\times$ for all i and $\vec{b}_i, \vec{b}_j = 0$ for $i \neq j$.

McDonald and Hershberger [4] proved the following theorem.

Theorem 2.2 (Theorem 3.2 of [4])**.** *Let (V,β) be an orthogonal space of rank $n \geq 2$. Then (V,β) processes an orthogonal basis $\mathcal{C}$ so that $[\beta]_{\mathcal{C}}$ is a diagonal matrix whose entries on the diagonal are units.*

Let (V,β) be an orthogonal space of rank $n \geq 2$. Let $\mathcal{C}$ be an orthogonal basis of V such that $[\beta]_{\mathcal{C}}$ is a diagonal matrix whose entries on the diagonal are units. From $[\beta]_{\mathcal{C}} = \operatorname{diag}(u_1,\dots,u_n)$ and u_i are units for all i. Assume that $u_1,\dots,u_r$ are squares and $u_{r+1},\dots,u_n$ are non-squares. Since $R^\times$ is a disjoint union of the cosets $(R^\times)^2$ and $z(R^\times)^2$ for some non-square unit z, we have $u_i = w_i^2$ for some $w_i \in R^\times$, $i = 1,\dots,r$ and $u_j = zw_j^2$ for some $w_j \in R^\times$, $j = r+1,\dots,n$. Thus, $[\beta]_{\mathcal{C}} = \operatorname{diag}(u_1,\dots,u_r) \oplus z\operatorname{diag}(w_{r+1},\dots,w_n)$ which is cogredient to $I_r \oplus zI_{n-r}$. If $n-r$ is even, Lemma 2.1 implies that $[\beta]_{\mathcal{C}}$ is cogredient to I_n. If $n-r$ is odd, then $n-r-1$ is even and so $[\beta]_{\mathcal{C}}$ is cogredient to $I_{n-1} \oplus (z)$ by the same lemma. Note that I_n and $I_{n-1} \oplus (z)$ are not cogredient since z is non-square. We record this result in the next theorem.

Theorem 2.3. *Let z be a non-square unit in R. Then $[\beta]_{\mathcal{C}}$ is cogredient to either I_n or $I_{n-1} \oplus (z)$.*

The next lemma follows by a simple calculation.

Lemma 2.4. *Let z be a non-square unit in R and and ν a positive integer. Write $H_{2\nu} = \begin{pmatrix} 0 & I_\nu \\ I_\nu & 0 \end{pmatrix}$.*

(1) If $-1 \in (R^\times)^2$, then I_ν is cogredient to $H_{2\nu}$ and $\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \approx \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}$.

(2) If $-1 \notin (R^\times)^2$, then $I_\nu \oplus zI_\nu$ is cogredient to $H_{2\nu}$ and $I_2 \approx \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}$.

Proof. First we observe that if $-1 = u^2$ for some unit u, then

$$\begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}.$$

However, if -1 is non-square, then $-1 = zc^2$ for some unit $c \in R$ and

$$\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -zc^2 \end{pmatrix} = I_2.$$

Next, a simple calculation with $P = \frac{1}{2}\begin{pmatrix} I_\nu & -I_\nu \\ I_\nu & I_\nu \end{pmatrix}$ shows that $L = 2\begin{pmatrix} I_\nu & 0 \\ 0 & -I_\nu \end{pmatrix}$ is cogredient to $H_{2\nu}$. Clearly, if -1 is square, L is cogredient to $I_{2\nu}$. Assume that -1 is non-square. By Proposition 1.2 (2), $-1 = zc^2$ for some unit c which also implies that 2 or -2 must be a square unit. If 2 is a square unit, then

$$L \approx I_\nu \oplus (-I_\nu) \approx I_\nu \oplus zc^2 I_\nu \approx I_\nu \oplus zI_\nu.$$

Similarly, if -2 is a square unit, then

$$L \approx (-I_\nu) \oplus I_\nu \approx zc^2 I_\nu \oplus I_\nu \approx I_\nu \oplus zI_\nu.$$

Therefore, $I_\nu \oplus zI_\nu$ is cogredient to $H_{2\nu}$. □

Next, we apply Lemmas 2.1 and 2.4 in the following calculations. We distinguish three cases. Let z be a non-square unit and ν a positive integer.

(1) Assume that -1 is square. Then
   (a) $I_{2\nu} \approx H_{2\nu}$ and $I_{2\nu+1} \approx H_{2\nu} \oplus (1)$.

(b) $I_{2\nu} \oplus (z) \approx H_{2\nu} \oplus (z)$ and $I_{2(\nu-1)} \oplus (z) \approx I_{2(\nu-1)} \oplus \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \approx H_{2\nu-1} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}$.

(2) Assume that -1 is non-square and ν is even. Then

(a) $I_{2\nu} \approx I_\nu \oplus I_\nu \approx I_\nu \oplus zI_\nu \approx H_{2\nu}$ and $I_{2\nu+1} \approx I_\nu \oplus I_\nu \oplus (1) \approx I_\nu \oplus zI_\nu \oplus (1) \approx H_{2\nu} \oplus (1)$.

(b) $I_{2\nu} \oplus (z) \approx I_\nu \oplus I_\nu \oplus (z) \approx I_\nu \oplus zI_\nu \oplus (z) \oplus H_{2\nu} \oplus (z)$ and

$$I_{2\nu-1} \oplus (z) \approx I_{\nu-2} \oplus I_{\nu-2} \oplus I_3 \oplus (z) \approx I_{\nu-2} \oplus zI_{\nu-2} \oplus I_3 \oplus (z)$$
$$\approx I_{\nu-1} \oplus zI_{\nu-1} \oplus I_2 \approx H_{2(\nu-1)} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}.$$

(3) Assume that -1 is non-square and ν is odd. Then

(a) $I_{2\nu} \approx I_{\nu-1} \oplus I_{\nu-1} \oplus I_2 \approx I_{\nu-1} \oplus zI_{\nu-1} \oplus I_2 \approx H_{2(\nu-1)} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -z \end{pmatrix}$ and
$I_{2\nu+1} \approx I_{\nu-1} \oplus I_{\nu-1} \oplus I_2 \oplus (1) \approx I_{\nu-1} \oplus zI_{\nu-1} \oplus zI_2 \oplus (1) \oplus I_\nu \oplus zI_\nu \oplus (z) \approx H_{2\nu} \oplus (z)$.

(b) $I_{2\nu} \oplus (z) \approx I_{\nu-1} \oplus I_{\nu-1} \oplus I_2 \oplus (z) \approx I_{\nu-1} \oplus zI_{\nu-1} \oplus I_2 \oplus (z) \approx I_\nu \oplus zI_\nu \oplus (1) \approx H_{2\nu} \oplus (1)$ and $I_{2\nu-1} \oplus (z) \approx I_{\nu-1} \oplus I_{\nu-1} \oplus (1) \oplus (z) \approx I_{\nu-1} \oplus I_{\nu-1} \oplus (1) \oplus (z) \approx I_\nu \oplus zI_\nu \approx H_{2\nu}$.

This proves a cogredient standard form of an orthogonal space over a finite local ring of odd characteristic.

Theorem 2.5. *Let R be a finite local ring of odd characteristic and let (V, β) be an orthogonal space where V is a free R-module of rank $n \geq 2$. Then there exists a $\delta \in \{0, 1, 2\}$ such that $\nu = \dfrac{n-\delta}{2} \geq 1$ and the associating matrix of β is cogredient to*

$$S_{2\nu+\delta,\Delta} = \begin{pmatrix} 0 & I_\nu & \\ I_\nu & 0 & \\ & & \Delta \end{pmatrix},$$

where

$$\Delta = \begin{cases} \emptyset (\textit{disappear}) & \textit{if } \delta = 0, \\ (1) \ \textit{or} \ (z) & \textit{if } \delta = 1, \\ \operatorname{diag}(1, -z) & \textit{if } \delta = 2, \end{cases}$$

and z is a fixed non-square unit of R.

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